Lecture 11

7.4 - Integration by Partial Fractions

Consider the integral $\int \sec x\,dx$. The trick used to compute it is a bit out of the blue...

Consider this:

$$\int \sec x\,dx = \int \frac{1}{\cos x}\,dx = \int \frac{\cos x}{\cos^2 x}\,dx = \int \frac{\cos x}{1-\sin^2 x}\,dx \quad \left(\begin{array}{l} u=\sin x \\ du=\cos x\,dx \end{array}\right)$$

$$= \int \frac{1}{1-u^2}\,du$$

How do we integrate this? Notice:

$$\frac{1}{1-u^2} = \frac{1}{(1+u)(1-u)} = \frac{1/2}{1+u} + \frac{1/2}{1-u} \qquad (\star)$$

So, $$\int \frac{1}{1-u^2}\,du = \frac{1}{2}\left(\int \frac{1}{1+u}\,du + \int \frac{1}{1-u}\,du \right)$$

$$= \frac{1}{2}\left(\ln|1+u| - \ln|1-u| \right) + C$$

$$= \frac{1}{2}\ln\left|\frac{1+u}{1-u}\right| + C = \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| + C$$

$$= \frac{1}{2}\ln\left|\frac{(1+\sin x)^2}{1-\sin^2 x}\right| + C = \ln\left|\sqrt{\frac{(1+\sin x)^2}{\cos^2 x}}\right| + C = \ln\left|\frac{1+\sin x}{\cos x}\right| + C$$

$$= \boxed{\ln|\sec x + \tan x| + C}$$

The procedure in (☆) is called Partial Fraction decomposition.

Fundamental Theorem of Algebra

Every polynomial can be factored into linear factors of the form $ax+b$ and irreducible quadratic factors of the form ax^2+bx+c (irred. means $b^2-4ac<0$).

Def: A rational function is a quotient of two polynomials

$$\frac{P(x)}{Q(x)}.$$

The goal of this section is to integrate rational functions. The procedure to integrate them has up to 3 steps:

1) If the degree of $P(x)$ is larger than that of $Q(x)$

2) Unless the rational part can be integrated in other ways (u-sub, trig. sub., etc.), use partial fractions to decompose it

3) Integrate.

The method of partial fractions will decompose a <u>proper</u> rational function $\frac{R(x)}{Q(x)}$ $(\deg R < \deg Q)$ as a sum of <u>partial fractions</u> of the form

$$\frac{A}{(ax+b)^k} \quad \text{and/or} \quad \frac{Ax+B}{(ax^2+bx+c)^k}$$

where A & B are constants and $k>0$.

We already know how to integrate these types of expressions.

We will go through increasingly more complex examples:

$\boxed{1.}$ $Q(x)$ is a product of distinct linear factors:

$$Q(x) = (a_1x+b_1)(a_2x+b_2)\cdots(a_nx+b_n)$$

$$\frac{R(x)}{Q(x)} = \frac{R(x)}{(a_1x+b_1)\cdots(a_nx+b_n)} = \frac{A_1}{a_1x+b_1} + \cdots + \frac{A_n}{a_nx+b_n}$$

Multiply through by the LCD (i.e. $Q(x)$):

$$R(x) = A_1(a_2x+b_2)\cdots(a_nx+b_n) + A_2(a_1x+b_1)(a_3x+b_3)\cdots(a_nx+b_n)$$
$$+ A_n(a_1x+b_1)\cdots(a_{n-1}x+b_{n-1})$$

Equating coefficients of powers of x, we can solve for $A_1, \cdots, A_n$.

Ex: Evaluate $\int \frac{1}{x^2-25}\,dx$

$$\frac{1}{x^2-25} = \frac{1}{(x+5)(x-5)} = \frac{A}{x+5} + \frac{B}{x-5} \Rightarrow 1 = A(x-5)+B(x+5)$$
$$= (A+B)x + (-5A+5B)$$

$$= \begin{cases} A+B=0 \Rightarrow A=-B \\ -5A+5B=1 \Rightarrow 10B=1 \Rightarrow B=\frac{1}{10} \Rightarrow A=\frac{-1}{10} \end{cases}$$

$$\int \frac{1}{x^2-25}\,dx = \int\left(\frac{-1/10}{x+5} + \frac{1/10}{x-5}\right)dx = \frac{1}{10}\left(-\ln|x+5| + \ln|x-5|\right) + C$$

$$= \boxed{\frac{1}{10}\ln\left|\frac{x-5}{x+5}\right| + C}$$

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2. $Q(x)$ has repeated linear factors, i.e., $(a_i x+b_i)^k$, $k>1$.

The idea is similar, but the difference is for every factor of the form $(a_i x+b_i)^k$ that shows up, we include a sum of the form $\frac{A_1}{a_i x+b_i} + \frac{A_2}{(a_i x+b_i)^2} + \cdots + \frac{A_k}{(a_i x+b_i)^k}$

For example, the decomposition of:

$$\frac{x^3+2x+2}{(x-2)^3(x-1)^2} = \frac{A_1}{x-2} + \frac{A_2}{(x-2)^2} + \frac{A_3}{(x-2)^3} + \frac{A_4}{(x-1)} + \frac{A_5}{(x-1)^2}$$

Ex: Compute $\int \frac{2x+3}{x^3+2x^2+x}\,dx$

$$\frac{2x+3}{x^3+2x^2+x} = \frac{2x+3}{x(x^2+2x+1)} = \frac{2x+3}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$\Rightarrow 2x+3 = A(x+1)^2 + Bx(x+1) + Cx$$
$$= (A+B)x^2 + (2A+B+C)x + A$$

$$\Rightarrow \begin{cases} A+B = 0 & ① \\ 2A+B+C = 2 & ② \\ A = 3 & ③ \end{cases}$$

$$③ \Rightarrow A = 3 \overset{①}{\Longrightarrow} B = -3 \overset{②}{\Rightarrow} C = 2-2(3)-(-3) = 2-6+3 = -1$$

$$\int \frac{2x+3}{x^3+2x^2+x}\,dx = \int \left(\frac{3}{x} - \frac{3}{x+1} - \frac{1}{(x+1)^2} \right) dx$$

$$= 3\ln|x| - 3\ln|x+1| + \frac{1}{x+1} + C$$

$$= \boxed{3\ln\left|\frac{x}{x+1}\right| + \frac{1}{x+1} + C}$$

$\boxed{3.}$ $Q(x)$ has irreducible quadratic factors, none of which are repeated. For every factor $a_ix^2+b_ix+c_i$ we include a term $\frac{A_ix+B_i}{a_ix^2+b_ix+c_i}$.

For example, the decomposition of:

$$\frac{x^2+x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

Ex: Evaluate $\int \frac{4x}{x^3+x^2+x+1}\,dx$

$$\frac{4x}{x^3+x^2+x+1} = \frac{4x}{(x^2+1)(x+1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+1}$$

$$\Rightarrow 4x = (Ax+B)(x+1) + C(x^2+1)$$
$$= (A+C)x^2 + (A+B)x + (B+C)$$

$$\Rightarrow \begin{cases} A + C = 0 & ① \Rightarrow A = -C \\ A + B = 4 & ② \\ B + C = 0 & ③ \Rightarrow B = -C \end{cases} \quad A = B$$

$$② \Rightarrow 2A = 4 \Rightarrow A = 2 \Rightarrow B = 2 \Rightarrow C = -2$$

$$\int \frac{4x}{x^3+x^2+x+1}\,dx = \int \left(\frac{2x+2}{x^2+1} - \frac{2}{x+1}\right) dx$$

$$= \underbrace{\int \frac{2x}{x^2+1}\,dx}_{\text{u-sub: } u=x^2+1} + 2\underbrace{\int \frac{1}{x^2+1}\,dx}_{\arctan x} - 2\underbrace{\int \frac{1}{x+1}\,dx}_{\ln|x+1|}$$

$$= \boxed{\ln|x^2+1| + 2\arctan x - 2\ln|x+1| + C}$$